# Computation of Very High Dimensional integrals by Quasi Monte-Carlo methods

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Let $\mathbf{X} = (X_1, \dots, X_n)$ a Gaussian vector of dimension $n$. Many problems of simultaneous statistical inference correspond to the computation of the probability of $\mathbf{X}$ to be in an hyper-rectangle.

The evaluation of such a probability for $n$ of the size of (say) 1000 seems very difficult. In can be conducted in two steps: the first uses elementary conditioning and the change of variable formula to transform the probability into an integral over the unit hyper-cube:

$$\int_{[0,1]^n} h(\mathbf{t}) d\mathbf{t}.$$

This transformation already implies an important reduction of variance if a Monte-Carlo (**MC**) method is used.

The second step uses an evaluation on the integral over the hyper-cube using a lattice rule that generate low discrepancy sequences.

More precisely, let $n$ be prime and let $\mathbf{Z}_1$ be a "nice integer sequence" in $\mathbb{N}^n$, the rule consist of choosing

$$\mathbf{t}_i = \left\{ \frac{i.\mathbf{z}}{M} \right\} \quad \text{and computing } \widehat{I} = \frac{1}{M} \sum_{i=1}^{M} h(\mathbf{t}_i)$$

where the notation $\{\}$ means that we have taken the fractional part componentwise. $M$ is chosen prime.

**Theorem 1** *(Nuyens and Cools, 2006) Assume that $h$ is the tensorial product of periodic functions that belong to a Koborov space (RKHS). Then the minimax sequence and the worst error can be calculated by a polynomial algorithm.*

This result concerns the "worst case" so in many cases, the convergence is faster. Numerical results show it is roughly $\mathcal{O}(M^{-1})$ thus much faster than **MC**.
If $h$ does not satisfies the conditions of the preceding theorem we can still hope **QMC** to be faster than **MC** and a reliable estimation the estimation error can

be obtained by adding a Monte-Carlo step:

Let $(t_i, i)$ be the lattice sequence, the way of estimating the integral can be turn to be random but exactly unbiased by setting

$$\widehat{I}(U) = 1/M \sum_{i=1}^{M} h\big(\{\mathbf{t}_i + U\}\big)$$

where $U$ is uniform on $[0,1]^n$. It is clear that $\mathbb{E} \oint \widehat{I}(U) = I$ and general considerations on QMC integration imply that $\widehat{I}(U)$ has **small variance**.

So we can make $N$ independent replications of this calculation, computing

$$\widehat{\widehat{I}} = 1/N(\widehat{I}(U_1) + \cdots + \widehat{I}(U_N)$$

and construct Student-type confidence intervals. This interval is correct whatever the properties of the function $h$ are. In practice $N$ is chosen rather small (12) so that the **MC** implies roughly a loss of speed of $\sqrt{12}$ with respect to a pure **QMC** method. But on the other hand we have a **reliable estimation of error**.

We will present some numerical applications and also application to design of experiments in large dimension making a comparison with LHS and Othogonal Arrays.

# References


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